Definition 1 *An $m \times n$ matrix A satisfies a* null-space property *of order k with constant C if for any $\eta \in \Re^n$ such that $A\eta = 0$, and any set $T \subset \{1 \dots n\}$ of size k, we have*

$$\|\eta\|_1 \leq C\|\eta_{-T}\|_1$$

where $-T$ denotes a complement of T.

Lemma 1 *Suppose that A satisfies RIP of order $(c+2)k$ with constant δ, $c > 1$. Then A satisfies the nullspace property of order $2k$ with constant $C = 1 + \sqrt{2/c}(1+\delta)/(1-\delta)$.*

Proof. Let T be the set of the largest (in absolute value) $2k$ coefficients of η. Let $T_0 = T$, T_1 be the set of indices of the next $M = ck$ largest coefficients of η. Let T_s be the last set of such indices. Also, let $\eta_0 = \eta_{T_0} + \eta_{T_1}$.

Since $A\eta = 0$, we have $A\eta_0 = -A(\eta_{T_2} + \dots \eta_{T_s})$. Therefore

$$\begin{aligned}
\|\eta_T\|_2 &\leq \|\eta_0\|_2 \\
&\leq (1-\delta)^{-1}\|A\eta_0\|_2 \\
&= (1-\delta)^{-1}\|A(\eta_{T_2} + \dots \eta_{T_s})\|_2 \\
&\leq (1-\delta)^{-1}\sum_{j=2}^{s}\|A\eta_{T_j}\|_2 \\
&\leq (1-\delta)^{-1}(1+\delta)\sum_{j=2}^{s}\|\eta_{T_j}\|_2
\end{aligned}$$

For any $i \in T_{j+1}$ and $l \in T_j$ we have $|\eta_i| \leq |\eta_j|$, and therefore $|\eta_i| \leq \|\eta_{T_j}\|_1/M$. It follows that

$$\|\eta_{T_{j+1}}\|_2 \leq (M(\|\eta_{T_j}\|_1/M)^2)^{1/2} = \|\eta_{T_j}\|_1/M^{1/2}$$

Therefore

$$\begin{aligned}
\|\eta_T\|_2 &\leq (1-\delta)^{-1}(1+\delta)/M^{1/2}\sum_{j=1}^{s}\|\eta_{T_j}\|_1 \\
&= (1-\delta)^{-1}(1+\delta)/M^{1/2}\|\eta_{-T}\|_1
\end{aligned}$$

Since $\|\eta_T\|_1/(2k)^{1/2} \leq \|\eta_T\|_2$, we get

$$\|\eta_T\|_1 \leq (2k)^{1/2}(1-\delta)^{-1}(1+\delta)/M^{1/2}\|\eta_{-T}\|_1$$

Therefore

$$\|\eta\|_1 = \|\eta_T\|_1 + \|\eta_{-T}\|_1 \leq [1 + (1-\delta)^{-1}(1+\delta)(2/c)^{1/2}]\|\eta_{-T}\|_1$$

Lemma 2 *Assume A satisfies the nullspace property of order $2k$ with constant $C < 2$. Then for x^* that minimizes $\|x^*\|_1$ subject to $Ax^* = Ax$ we have*

$$\|x - x^*\|_1 \leq \frac{2C}{2-C} Err_1^k(x)$$

Proof Let $\eta = x^* - x$. and let T be the set of k largest coefficients of x. From the null-space property we have

$$\|\eta_T\|_1 \leq (C-1)\|\eta_{-T}\|_1 \tag{1}$$

Since x^* is the minimizer, we have $\|x^*\|_1 \leq \|x\|_1$, which we rewrite as

$$\|x^*_T\|_1 + \|x^*_{-T}\|_1 \leq \|x_T\|_1 + \|x_{-T}\|_1$$

It follows that

$$\|x_T\|_1 - \|\eta_T\|_1 - \|x_{-T}\|_1 + \|\eta_{-T}\|_1 \leq \|x_T\|_1 + \|x_{-T}\|_1$$

and therefore

$$\|\eta_{-T}\|_1 \leq \|\eta_T\|_1 + 2\|x_{-T}\|_1 = \|\eta_T\|_1 + 2Err_1^k(x)$$

From Eq. 1 we have

$$\|\eta_{-T}\|_1 \leq (C-1)\|\eta_{-T}\|_1 + 2Err_1^k(x)$$

which implies

$$\|\eta_{-T}\|_1 \leq \frac{2}{2-C} Err_1^k(x)$$

Thus

$$\|\eta\|_1 \leq C\|\eta_{-T}\|_1 \leq \frac{2C}{2-C} Err_1^k(x)$$